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Realisations of the real semisimple Lie algebras: a method of construction

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Abstract. For an arbitrary real semisimple Lie algebra g, a method is presented of constructing skew-Hermitian realisations in terms of quantum canonical variables p_i, q_i. The construction starts with a decomposition $g = n_+^b \oplus g_0^b \oplus n_-^b$ of g, which is a simple generalisation of the triangle decomposition, employing substantially an induced representation of g with respect to a suitable representation σ of the subalgebra $g_0^b \oplus n_-^b$. It is shown that the realisations obtained are Schurean. As examples, the realisations for the algebras sp(2, R) and gl(3, R) are calculated explicitly.

1. Introduction

A realisation (also, canonical realisation or boson representation) of a Lie algebra g denotes an expression of the elements of g by means of polynomials in quantum canonical variables p_i, q_i which preserve the commutation relations of g. Several types of boson representations are used in physical applications; e.g., see Holstein and Primakoff (1940), Dyson (1956), Schwinger (1965). The number of publications on this subject has increased rapidly in the last few years in connection with the introduction of the interacting boson model in nuclear physics; see Dobaczewski (1981a, b, 1982) or Klein (1983) and references therein. Deenen and Quesne (1982a, b, 1984a, b) have given explicit forms of realisations for the symplectic algebras sp($2d$, R). These realisations are physically interesting in connection with a microscopic model for the system of N nucleons in $d(=1, 2$, or $3)$ dimensions; for a particular survey of this model see Moshinsky (1984 and references therein). In the recent work of Dündarer and Gürsey (1984) a class of realisations of the algebra su(3) was constructed. Canonical realisations also have many applications in representation theory—see Baird and Biedenharn (1963), Kihlberg (1965), Barut and Raczka (1977) or Burdík *et al* (1981a, b).

In the papers Havlíček and Exner (1975a, b, 1978) and by Havlíček and Lassner (1975, 1976a, b, c, 1977) extensive families of realisations for all complex classical Lie algebras and for most of their real forms were constructed. All these realisations have two interesting properties with respect to an application in the representation theory. They are Schurean (i.e. they have the Casimir operators realised by multiples of unity), and in the case of real forms, they are skew-Hermitian.

In this paper we present an algebraic method of constructing realisations for any real semisimple Lie algebra g. It is shown that any induced representation of g can be rewritten as a boson one. This fact is the starting point of our construction. The resulting realisations are Schurean and skew-Hermitian. The Lie algebras sp(2, R) and gl(3, R) are treated as illustrative examples. In the case of the algebra sp(2, R), we get the Dyson realisations which were constructed by Deenen and Quesne (1984b). The realisations of the algebra gl(3, R) are of the same type as the realisation obtained by Havlíček and Lassner (1975). It is likely that application of the present method to classical simple Lie algebras will lead to realisations similar to those given explicitly in the papers mentioned. However, we expect our method to give other realisations which are not obtained. Some positive indications have been obtained already; the results will be presented in a forthcoming series of papers.

2. Preliminaries

Here we shall briefly review some notions needed in the following.

Let g be a real or complex Lie algebra. By $\tilde{g}$ we denote its complexification; then $g=\tilde{g}$ if g is complex.

Let g and g_0 be Lie algebras, further let $U(\tilde{g})$ and $U(\tilde{g}_0)$ be the enveloping algebras of their complexifications, and finally, let W_{2n} be the complex Weyl algebra in n-canonical pairs $p_i, q_i, i=1,2,\ldots,n$ which fulfil the usual canonical commutation relations

$$[p_i, p_j]=[q_i, q_j]=0, \qquad [p_i, q_j]=\delta_{ij}1.$$

A realisation of a Lie algebra g is a homomorphism

$$\tau: g \rightarrow W_{2n} \otimes U(\tilde{g}_0).$$

For more details about the definition and properties of the canonical realisations see Exner *et al* (1976).

The homomorphism τ extends naturally to the homomorphic mapping (denoted by the same symbol τ) of the enveloping algebra $U(\tilde{g})$ into $W_{2n}\otimes U(\tilde{g}_0)$. Let $Z(\tilde{g})$ be the centre of $U(\tilde{g})$. A realisation τ is called Schurean or a Schur-realisation if all central elements $C\in Z(\tilde{g})$ are realised by $1\otimes C_0$ where the C_0's are central elements of the enveloping algebra $U(\tilde{g}_0)$.

Recall that the involution on an associative algebra g is the mapping '+': $\tilde{g}\rightarrow\tilde{g}$ obeying the relations

$$(\alpha X+\beta Y)^+=\bar{\alpha}X^+ +\bar{\beta}Y^+$$
$$(XY)^+=Y^+X^+, \qquad (X^+)^+=X.$$

Let g_0 be a real Lie algebra. An involution on W_{2n} together with an involution on the enveloping algebra $U(\tilde{g}_0)$ defines naturally an involution on $W_{2n}\otimes U(\tilde{g}_0)$ by

$$\left(\sum_j \alpha_j \Pi_j \otimes g_j\right)^+ \equiv \sum_j \bar{\alpha}_j \Pi_j^+ \otimes g_j^+.$$

In what follows, we consider this involution determined by the involutions on W_{2n} and $U(\tilde{g}_0)$ respectively, which are generated by the relations

$$(q_i)^+=-q_i \qquad (p_i)^+=p_i \tag{1a, b}$$

on the algebra W_{2n}, and

$$Y^+ = -Y, \qquad Y \in \mathrm{g}_0 \tag{1c}$$

on the algebra $U(\tilde{\mathrm{g}}_0)$.

Let g be a real Lie algebra and let '+' be an involution on $W_{2n} \otimes U(\tilde{\mathrm{g}}_0)$ described above. A realisation of g on $W_{2n} \otimes U(\tilde{\mathrm{g}}_0)$ is called skew-Hermitian, if all elements $X \in \mathrm{g}$, obey the relation

$$(\tau(X))^+ = -\tau(X).$$

3. The general construction

The construction presented in this section is based on the method of induced representations of Lie algebras. We suppose that the reader is familiar with the basic notions of the theory of the induced representations, whose exposition can be found, e.g., in Dixmier (1974).

Let g′ be a Lie subalgebra of g and g_0 be a Lie subalgebra of g′, i.e. $\mathrm{g}_0 \subset \mathrm{g}' \subset \mathrm{g}$. In the following, we shall use a basis $X_1, X_2, \ldots, X_m$ in the algebra g, where $X_{n+1}, X_{n+2}, \ldots, X_s$ is a basis for g_0 and $X_{n+1}, X_{n+2}, \ldots, X_m$ is a basis for g′. Furthermore, we shall assume that an auxiliary representation σ of the algebra g′ on the space V is given such that

$$\sigma(X_{s+j}) = 0 \qquad j = 1, 2, \ldots, m-s \tag{2a}$$

$$\sigma|\mathrm{g}_0 \text{ is faithful.} \tag{2b}$$

On the space $U(\tilde{\mathrm{g}}) \otimes V$, we can simply define the representation l of g by

$$l(X)(u \otimes v) = Xu \otimes v$$

for all $X \in \mathrm{g}$, $u \in U(\tilde{\mathrm{g}})$ and $v \in V$. This representation is called the left regular representation.

Consider the subspace L in $U(\tilde{\mathrm{g}}) \otimes V$ generated by

$$(uy \otimes v) - (u \otimes \sigma(y)v), \qquad u \in U(\tilde{\mathrm{g}}), yU(\tilde{\mathrm{g}}) \text{ and } v \in V.$$

It is easy to see that L is invariant with respect to l. Hence we may define quotient representation of g on the space $W = (U(\tilde{\mathrm{g}}) \otimes V)/L$. This representation is called induced representation and is denoted by ind(g, σ). If $\{v_j\}$ is a basis in the space V, then the vectors

$$|\tilde{k}\rangle \otimes v_j \equiv |k_1, \ldots, k_n\rangle \otimes v_j \equiv X_1^{k_1} \ldots X_n^{k_n} \otimes v_j$$

$k_1, k_2, \ldots, k_n \in N_0$, where N_0 is the set of all non-negative integers, form a basis in W.

We define the creation and annihilation operators $\bar{a}_i$, a_i, $i = 1, 2, \ldots, n$ and the operators $\tilde{X}_r$, $r = n+1, \ldots, s$ in the following way:

$$\bar{a}_i|k_1, \ldots, k_i, \ldots, k_n\rangle \otimes v \equiv |k_1, \ldots, k_i + 1, \ldots, k_n\rangle \otimes v \tag{3a}$$

$$a_i|k_1, \ldots, k_i, \ldots, k_n\rangle \otimes v \equiv k_i|k_1, \ldots, k_i - 1, \ldots, k_n\rangle \otimes v \tag{3b}$$

(notice the normalisation convention), and

$$\tilde{X}_r|k_1, \ldots, k_n\rangle \otimes v \equiv |k_1, \ldots, k_n\rangle \otimes \sigma(X_r)v. \tag{3c}$$

Obviously they obey the commutation relations

$$[a_i, a_j] = [\bar{a}_i, \bar{a}_j] = 0, \qquad [a_i, \bar{a}_j] = \delta_{ij} 1$$
$$[a_i, \tilde{X}_r] = [\bar{a}_i, \tilde{X}_r] = 0.$$

Theorem 1. Let $\rho \equiv \mathrm{ind}(\mathrm{g}, \sigma)$, then all the operators $\rho(X)$, $X \in \mathrm{g}$, can be rewritten in the form

$$\rho(X) = \sum_{i=1}^{n} \bar{a}_i \alpha_i^X \otimes 1 + \sum_{r=n+1}^{s} \alpha_r^X \otimes \tilde{X}_r \tag{4}$$

where α_t^X, $t = 1, 2, \ldots, s$ are, in general, the infinite sums of monomials in the operators $a_1, \ldots, a_n$.

Proof. The formula

$$XY^k = \sum_{i=0}^{k} \binom{k}{i} Y^{k-i}[\ldots[[X, Y], Y],] \ldots, Y] \qquad i \text{ times} \tag{5}$$

implies for $XX_1^{k_1} \ldots X_n^{k_n}$ the following equality

$$\begin{aligned} XX_1^{k_1} \ldots X_n^{k_n} = & \sum_{j=1}^{n} \sum_{\tilde{i}=\tilde{0}}^{\tilde{k}} c_{j\tilde{i}}^X \binom{\tilde{k}}{\tilde{i}} X_1^{k_1-i_1} \ldots X_j^{k_j-i_j+1} \ldots X_n^{k_n-i_n} \\ & + \sum_{r=n+1}^{m} \sum_{\tilde{i}=\tilde{0}}^{\tilde{k}} c_{r\tilde{i}}^X \binom{\tilde{k}}{\tilde{i}} X_1^{k_1-i_1} \ldots X_n^{k_n-i_n} X_r \end{aligned} \tag{6}$$

where $c_{t\tilde{i}}^X$, $t = 1, 2, \ldots, m$, are constants independent of $\tilde{k}$ and

$$\binom{\tilde{k}}{\tilde{i}} = \binom{k_1}{i_1} \ldots \binom{k_n}{i_n}.$$

Of course, $\tilde{0}$ means $(0, \ldots, 0)$. For any $\tilde{k}$, we define

$$^{\tilde{k}}\alpha_t^X = \sum_{\tilde{i}=\tilde{0}}^{\tilde{k}} c_{t\tilde{i}}^X a^{\tilde{i}} \qquad \text{where } a^{\tilde{i}} \equiv a_1^{i_1} \ldots a_n^{i_n}. \tag{7}$$

According to the definition ($3a$) and ($3b$), $a^{\tilde{k}'}(|\tilde{k}\rangle \otimes v) = 0$ for any $\tilde{k}'$ such that $\Sigma_{i=1}^n k_i' > \Sigma_{i=1}^n k_i$. This implies the equality $^{\tilde{k}}\alpha_t^X(|\tilde{k}\rangle \otimes v) = {}^{\tilde{k}}\alpha_t^X(|\tilde{k}\rangle \otimes v)$ if $\Sigma_{i=1}^n k_1' > \Sigma_{i=1}^n k_i$, which further means that the infinite sums

$$\alpha_t^X = \sum_{\tilde{i}=\tilde{0}} c_{t\tilde{i}}^X a^{\tilde{i}} \tag{8}$$

are well defined. It is obvious that $\alpha_t^X(|\tilde{k}\rangle \otimes v) = {}^{\tilde{k}}\alpha_t^X(|\tilde{k}\rangle \otimes v)$. For any $(|\tilde{k}\rangle \otimes v) \in W$, $\rho(X)(|\tilde{k}\rangle \otimes v) = (XX_1^{k_1} \ldots X_n^{k_n}) \otimes v$ and according to ($3a$)-($3c$), (6), (7) and (8) the RHS further equals

$$\begin{aligned} & \left(\sum_{j=1}^{n} \bar{a}_j {}^{\tilde{k}}\alpha_t^X \otimes 1 + \sum_{r=n+1}^{s} {}^{\tilde{k}}\alpha_t^X \otimes \tilde{X}_r \right) (|\tilde{k}\rangle \otimes v) \\ & \qquad = \left(\sum_{j=1}^{n} \bar{a}_j \alpha_j^X \otimes 1 + \sum_{r=n+1}^{s} \alpha_r^X \otimes \tilde{X}_r \right) (|\tilde{k}\rangle \otimes v). \end{aligned}$$

This proves the theorem.

We now give conditions on $g' \subset g$ under which α_t^X, $t = 1, 2, \ldots, s$ defined by (8) are polynomials. Recall that this is required if one wants to obtain realisations in the sense of our definition. Let S be the subspace in g which is spanned by $X_1, \ldots, X_n$. For any $Y \in g$, we define by induction $M_i^Y = [S, M_{i-1}^Y]$, $M_0^Y = \mathbb{C}\{Y\}$.

Proposition 1. If there exists $k \in N_0$ such that $M_k^Y = \{0\}$ for any $Y \in g$, then α_t^Y, $t = 1, 2, \ldots, m$, are finite polynomials.

Proof. Obviously, if $\Sigma_{i=1}^n j_i > k$, then $c_{tj}^X = 0$ for any $X \in g$, $t = 1, 2, \ldots, m$, and according to the definition (8), the expressions of α_t^Y become finite polynomials.

Now the realisations sought are obtained easily by replacing operators in the above expressions by suitable algebraic objects. The mapping

$$\varphi(p_i) = a_i \tag{9a}$$

$$\varphi(q_i) = \bar{a}_i \qquad i = 1, 2, \ldots, n \tag{9b}$$

$$\varphi(X_r) = X_r \qquad r = n+1, \ldots, s \tag{9c}$$

extends naturally to a faithful representation $W_{2n} \otimes U(g_0)$ on W. Thus there exist φ^{-1}, and if α_j^X, α_r^X in (6) are polynomials, then the mapping τ:

$$\tau(X) = \varphi^{-1} \circ \rho(X)$$

is consistently defined. We denote $\beta_j^X = \varphi^{-1} \circ \alpha_j^X$.

Proposition 2. τ is a realisation of the algebra g in $W_{2n} \otimes U(\tilde{g}_0)$.

Proof.

$$\begin{aligned}\tau([X, Y]) &= \varphi^{-1} \circ \rho([X, Y]) = \varphi^{-1}([\rho(X), \rho(Y)]) \\ &= [\varphi^{-1} \circ \rho(X), \varphi^{-1} \circ \rho(Y)] = [\tau(X), \tau(Y)].\end{aligned}$$

Let τ be a realisation of the real algebra g in $W_{2n} \otimes U(\tilde{g}_0)$, which is of the type

$$\tau(X) = \sum_{j=1}^{n} q_j \beta_j^X \otimes 1 + \sum_{r=n+1}^{s} \beta_r^X \otimes X_r, \qquad X_r \in g_0, \tag{10}$$

where β_t^X are real polynomials in the variables $p_1, \ldots, p_n$. Then the following expressions are well defined

$$\tau'(X) = \tau(X) + \frac{1}{2} \sum_{j=1}^{n} \frac{\partial \beta_j^X}{\partial p_j} \otimes 1. \tag{11}$$

Theorem 2. τ' is a skew-Hermitian realisation of g in $W_{2n} \otimes U(\tilde{g}_0)$.

Proof. Using the relations $[\beta_j^X, q_i] = \partial \beta_j^X / \partial p_i$ and (1a)–(1c), one can check easily the skew-Hermiticity.

$$(\tau'(X))^+ = \left(\sum_{j=1}^{n} q_j \beta_j^X \otimes 1 + \sum_{r=n+1}^{s} \beta_r^X \otimes X_r + \frac{1}{2} \sum_{j=1}^{n} \frac{\partial \beta_j^X}{\partial p_j} \otimes 1 \right)^+$$

$$= -\sum_{j=1}^{n} \beta_j^X q_j \otimes 1 - \sum_{r=n+1}^{s} \beta_r^X \otimes X_r + \frac{1}{2}\sum_{j=1}^{n} \frac{\partial \beta_j^X}{\partial p_j} \otimes 1$$

$$= -\tau'(X).$$

Further

$$[\tau(X), \tau(Y)] = \left(\sum_{j=1}^{n} q_j \beta_j^X \otimes 1, \sum_{k=1}^{n} q_k \beta_k^Y \otimes 1\right) + P$$

$$= \left[\sum_{j=1}^{n} q_j \left(\sum_{k=1}^{n} \frac{\partial \beta_j^X}{\partial p_k} \beta_k^Y - \frac{\partial \beta_j^Y}{\partial p_k} \beta_k^X\right)\right] \otimes 1 + P$$

where the P is independent of q_i, $i = 1, 2, \ldots, n$. Since $[\tau(X), \tau(Y)] = \tau([X, Y])$, then according to formula (10)

$$\beta_j^{[X,Y]} = \sum_{k=1}^{n} \left(\frac{\partial \beta_j^X}{\partial p_k} \beta_k^Y - \frac{\partial \beta_j^Y}{\partial p_k} \beta_k^X\right). \tag{12}$$

Now a direct calculation gives

$$[\tau'(X), \tau'(Y)] = [\tau(X), \tau(Y)] + \left(\sum_{k=1}^{n} q_k \beta_k^X, \frac{1}{2}\sum_{j=1}^{n} \frac{\partial \beta_j^Y}{\partial p_j}\right) \otimes 1$$

$$+ \left(\frac{1}{2}\sum_{j=1}^{n} \frac{\partial \beta_j^X}{\partial p_j}, \sum_{k=1}^{n} q_k \beta_k^Y\right) \otimes 1$$

$$= \tau([X, Y]) - \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial^2 \beta_j^Y}{\partial p_j\, \partial p_k} \beta_k^X - \frac{\partial^2 \beta_j^X}{\partial p_j\, \partial p_k} \beta_k^Y\right) \otimes 1$$

$$= \tau([X, Y]) + \frac{1}{2}\sum_{j=1}^{n} \frac{\partial \beta_j^{[X,Y]}}{\partial p_j} \otimes 1 = \tau'([X, Y]).$$

Hence τ' is a realisation of g, and the proof is completed.

4. The construction for the real semisimple Lie algebra

In this section, we apply the general construction described above to the case of a real semisimple Lie algebra. Suppose that g is a real semisimple Lie algebra and that $b \in \mathrm{g}$ and $B = \{X_1, \ldots, X_n, X_{n+1}, \ldots, X_s, X_{s+1}, \ldots, X_{s+n}\}$ is a basis in g for which

$$[b, X_j] = \gamma_j X_j \tag{13a}$$

$$[b, X_{s+j}] = -\gamma_j X_{s+j} \tag{13b}$$

where $\gamma_j > 0$ for any $j = 1, 2, \ldots, n$ and further

$$[b, X_r] = 0, \qquad r = n+1, \ldots, s. \tag{13c}$$

The triangle decomposition of g implies the existence of such a basis and b in g. For details see Dixmier (1974, ch 1). In the example gl(3, R) we give such a basis explicitly and for other real forms of classical Lie algebras we will construct explicitly these bases in a forthcoming series of papers.

The element b and the basis B define for g the following direct decomposition

$$\mathrm{g} = \mathrm{n}_+^b \oplus \mathrm{g}_0^b \oplus \mathrm{n}_-^b$$

where

$$\mathrm{n}_{+}^{b} \equiv \mathbb{R}\{X_j, j=1,\ldots,n\} \tag{14a}$$

$$\mathrm{n}_{-}^{b} \equiv \mathbb{R}\{X_{s+j}, j=1,\ldots,n\} \tag{14b}$$

$$\mathrm{g}_{0}^{b} \equiv \mathbb{R}\{X_r, r=n+1,\ldots,s\}. \tag{14c}$$

The following lemma results directly from the Jacobi identity, the formulae (13*a*)-(13*c*) and the definitions (14*a*)-(14*c*).

Lemma 1.
(i) n_{+}^{b}, n_{-}^{b} are subalgebras of g.
(ii) $[\mathrm{n}_{+}^{b}, \mathrm{g}_{0}^{b}] \subset \mathrm{n}_{+}^{b}$, $[\mathrm{n}_{-}^{b}, \mathrm{g}_{0}^{b}] \subset \mathrm{n}_{-}^{b}$.
(iii) If $M_i^Y = [\mathrm{n}_{+}^{b}, M_{i-1}^Y]$, $M_0^Y = \mathbb{C}\{Y\}$ then there exists a k such that for any $i > k$ and $Y \in \mathrm{g}$, $M_i^Y = \{0\}$.

We now specify the g′, g_0 described at the beginning of § 3 in this way:

$$\mathrm{g}_0 \equiv \mathrm{g}_0^b \qquad \mathrm{g}' \equiv \mathrm{g}_0^b \oplus \mathrm{n}_{-}^{b}$$

and further the auxiliary representation σ fulfils

$$\sigma(\mathrm{g}_0^b) \text{ is faithful} \tag{15a}$$

$$\sigma|\mathrm{n}_{-}^{b} = 0. \tag{15b}$$

According to the assertion (iii) in lemma 1 we can construct the realisations τ_b and τ_b' by means of the method from § 3. These have the following property.

Theorem 3. τ_b and τ_b' are Schur realisations of g in the $\mathrm{W}_{2n} \otimes \mathrm{U}(\tilde{\mathrm{g}}_0^b)$.

Proof. First we prove one proposition which specifies forms of elements $Z(\tilde{\mathrm{g}})$.

Proposition 3. Any $C \in Z(\tilde{\mathrm{g}})$ can be written in the form

$$C = \sum_{\tilde{n}, \tilde{n}' \neq 0} X_1^{n_1} \ldots X_n^{n_n} Y_{\tilde{n},\tilde{n}'} X_{s+1}^{n_1'} \ldots X_{s+n}^{n_n'} + C_0 \tag{16}$$

where $Y_{\tilde{n},\tilde{n}'} \in \mathrm{U}(\tilde{\mathrm{g}}_0^b)$ and $C_0 \in Z(\tilde{\mathrm{g}}_0^b)$.

Proof. Write $C \in Z(\tilde{\mathrm{g}})$ in the form

$$C = \sum_{\substack{\tilde{n}, \tilde{n}'=0 \\ \tilde{n}, \tilde{n}' \neq 0}} X_1^{n_1} \ldots X_n^{n_n} Y_{\tilde{n},\tilde{n}'} X_{s+1}^{n_1'} \ldots X_{s+n}^{n_n'} + C_0 \tag{17}$$

where $Y_{\tilde{n},\tilde{n}'} \in \mathrm{U}(\tilde{\mathrm{g}}_0^b)$ and $C_0 \in \mathrm{U}(\tilde{\mathrm{g}}_0^b)$. The facts, that the element C commutes with b and the formulae (13*a*)-(13*c*) imply that, if

$$\sum_{i=1}^{n} n_i \gamma_i \neq \sum_{i=1}^{n} n_i' \gamma_i \qquad \text{then } Y_{\tilde{n},\tilde{n}'} = 0.$$

Since $\gamma_i > 0$ then, if $Y_{\tilde{n},\tilde{n}'} \neq 0$ $\Sigma_{i=1}^{n} n_i > 0$ also $\Sigma_{i=1}^{n} n_i' > 0$ and, therefore, the summation in (17) runs only over $\tilde{n}$, $\tilde{n}'$ for which $\tilde{n}, \tilde{n}' \neq \tilde{0}$. This and the condition (ii) in lemma 1 imply that if $X \in \tilde{\mathrm{g}}_0^b$, then $[X, C_0] = 0$. This completes the proof of proposition 3.

Using (16) and (15b) and the definition of induced representation we now calculate explicitly the operator $\rho(C)$.

$$\begin{aligned}\rho(C)(|\tilde{k}\rangle\otimes v) &= \rho(C)(\rho(X_1))^{k_1}\ldots(\rho(X_n))^{k_n}(|\tilde{0}\rangle\otimes v)\\ &= (\rho(X_1))^{k_1}\ldots(\rho(X_n))^{k_n}\rho(C)(|\tilde{0}\rangle\otimes v) = |\tilde{k}\rangle\otimes\sigma(C_0)v.\end{aligned}$$

This proves the Schur property of τ_b, because $\tau(C)=\varphi^{-1}\circ\rho(C)=1\otimes C_0$. Now we are going to check the same property for τ'_b. We define $\rho'(X)\equiv\varphi\circ\tau'(X)$. Before continuing the proof of theorem 3 we shall prove the following lemma.

Lemma 2.

(i) $\rho'(X_j)=\rho(X_j)$ holds for any $j=1,2,\ldots,n$.

(ii) For any X_r, $r=n+1,\ldots,s$ $\rho(X_r)=\sum_{j=1}^n \bar{a}_j\alpha\alpha_{\tilde{j}}^{X}\otimes 1+1\otimes\tilde{X}_r$.

(iii) For any X_r, $r=n+1,\ldots,s$ $\rho'(X_r)(|\tilde{0}\rangle\otimes v)=|\tilde{0}\rangle\otimes(c_r+\sigma(X_r))v$, where $c_r\in\mathbb{C}$.

(iv) $\rho'(X_{s+j})(|\tilde{0}\rangle\otimes v)=0$ holds for any $j=1,2,\ldots,n$.

Proof.

(i) Since γ_j, γ_i are positive, $\gamma_j+\gamma_i\neq\gamma_i$ holds for any $j,i=1,2,\ldots,n$. Therefore $c_{i\tilde{f}}^{X_k}=0$ for $f_i>0$, and further according to the definition (8) $\frac{1}{2}\sum_{i=1}^n \partial\alpha_i^{X_k}/\partial a_i=0$ for any X_k, $k=1,2,\ldots,n$ and also $\rho'(X_k)=\rho(X_k)$, $k=1,2,\ldots,n$.

(ii) The assertions (i) and (ii) from lemma 1 imply $c_{t\tilde{f}}^{X_r}=0$, $t=n+1,\ldots,s$ and $\tilde{f}$ for which $\sum_{i=1}^n f_i>0$ and, obviously, $c_{s\tilde{0}}^{X_r}=0$ for $s\neq r$. This gives the assertion directly according to definition (8).

(iii) By a direct calculation we obtain

$$\begin{aligned}\rho'(X_r)(|\tilde{0}\rangle\otimes v) &= \left(\rho(X_r)+\frac{1}{2}\sum_{j=1}^n\frac{\partial\alpha_j^{X_r}}{\partial a_j}\otimes 1\right)(|\tilde{0}\rangle\otimes v)\\ &= (\rho(X_r)+c_r)(|\tilde{0}\rangle\otimes v)=|\tilde{0}\rangle\otimes(c_r+\sigma(X_r))v.\end{aligned}$$

(iv) As in (i), any two γ_j, γ_i fulfil $\gamma_j-\gamma_i\neq\gamma_j$. This implies that $C_{i\tilde{I}_i}^{X_{s+k}}=0$ for any $k,i=1,2,\ldots,n$, where $\tilde{I}_i=(0,\ldots,0,1,0,\ldots,0)$ with 1 on ith place. This and (15b) give that

$$\rho'(X_{s+k})(|\tilde{0}\rangle\otimes v)=\left(\rho(X_{s+k})+\frac{1}{2}\sum_{i=1}^n\frac{\partial\alpha_i^{X_{s+k}}}{\partial a_i}\otimes 1\right)(|\tilde{0}\rangle\otimes v)=0.$$

Using the lemma just proved, we shall now evaluate $\rho(C)$ explicitly

$$\begin{aligned}\rho'(C)(|\tilde{k}\rangle\otimes v) &= \rho'(C)(\rho(X_1))^{k_1}\ldots(\rho(X_n))^{k_n}(|\tilde{0}\rangle\otimes v)\\ &= \rho'(C)(\rho'(X_1))^{k_1}\ldots(\rho'(X_n))^{k_n}(|\tilde{0}\rangle\otimes v)\\ &= (\rho(X_1))^{k_1}\ldots(\rho(X_n))^{k_n}\rho'(C_0)(|\tilde{0}\rangle\otimes v)=|\tilde{k}\rangle\otimes\sigma(C'_0)v\end{aligned}$$

where $C'_0\in U(\tilde{g}_0^b)$. Further we prove that $C'_0\in Z(\tilde{g}_0^b)$. Let X_r be any element of $\tilde{g}_0^b$, then

$$\begin{aligned}&[1\otimes C'_0, 1\otimes X_r]\\ &\quad=\left[1\otimes C'_0, \left(\sum_{j=1}^n q_j\beta_j^{X_r}+\frac{1}{2}\sum_{j=1}^n\frac{\partial\beta_j^{X_r}}{\partial p_j}\right)\otimes 1+1\otimes X_r\right]\\ &\quad=\varphi^{-1}[\rho'(C),\rho'(X_r)]=0.\end{aligned}$$

This completes the proof of the theorem.

5. The realisations for the algebras sp(2, *R*) and gl(3, *R*)

Now we are going to illustrate the construction by the examples of the algebras sp(2, R) and gl(3, R).

5.1. *The example of* sp(2, *R*)

The algebra sp(2, R) is the algebra of linear canonical transformations in one dimension. In the basis $\{D_{11}^+, E_{11}, D_{11}\}$, which was used by Deenen and Quesne (1984b) the generators satisfy the following commutation relations

$$[E_{11}, D_{11}^+]=2D_{11}^+ \qquad [E_{11}, D_{11}]=-2D_{11} \qquad [D_{11}, D_{11}^+]=4E_{11}. \tag{18}$$

Now we put $b\equiv E_{11}$ and $B\equiv\{D_{11}^+, E_{11}, D_{11}\}$, then using the formulae (13a)-(13c) and (14a)-(14c) we obtain

$$\mathfrak{n}_+^b=\mathbb{R}\{D_{11}^+\} \qquad \mathfrak{g}_0^b=\mathbb{R}\{E_{11}\} \qquad \mathfrak{n}_-^b=\mathbb{R}\{D_{11}\}.$$

The formulae (5) and (18) imply

$$D_{11}^+(D_{11}^+)^k=(D_{11}^+)^{k+1}$$
$$E_{11}(D_{11}^+)^k=2k(D_{11}^+)^k+(D_{11}^+)^kE_{11}$$
$$D_{11}(D_{11}^+)^k=4k(k-1)(D_{11}^+)^{k-1}+4k(D_{11}^+)^{k-1}E_{11}+(D_{11}^+)^kD_{11}.$$

According to the definitions (8) and (11) the realisations and then expressed explicitly by the relations

$$\tau(D_{11}^+)=q\otimes 1$$
$$\tau(E_{11})=2qp\otimes 1+1\otimes E_{11}$$
$$\tau(D_{11})=4qp^2\otimes 1+4p\otimes E_{11}$$

and

$$\tau'(D_{11}^+)=q\otimes 1$$
$$\tau'(E_{11})=2qp\otimes 1+1\otimes(E_{11}+1)$$
$$\tau'(D_{11})=4qp^2\otimes 1+4p\otimes(E_{11}+\tfrac{1}{2}).$$

These realisations are of Dyson type and coincide with those constructed by Deenen and Quesne (1984b) in the framework of partially coherent states.

5.2. *The example of* gl(3, *R*)

The algebra gl(3, R) is nine-dimensional with the standard basis $\{E_{ij}\colon i, j=1, 2, 3\}$, the elements of which obey

$$[E_{ij}, E_{kl}]=\delta_{jk}E_{il}-\delta_{il}E_{kl}. \tag{19}$$

If we denote $b\equiv E_{11}$ and $B\equiv\{E_{12}, E_{13}, E_{11}, E_{22}, E_{33}, E_{23}, E_{32}, E_{21}, E_{31}\}$ we obtain according to (13a)-(13c) and definitions (14a)-(14c)

$$\mathfrak{n}_+^b=\mathbb{R}\{E_{12}, E_{13}\}$$
$$\mathfrak{g}_0^b=R\{E_{11}, E_{22}, E_{33}, E_{23}, E_{32}\}$$
$$\mathfrak{n}_-^b=R\{E_{21}, E_{31}\}.$$

The explicit form of the realisations follows from the equalities (6) for $XE_{12}^{k_{12}}E_{13}^{k_{13}}$ with an arbitrary $X \in B$, which are calculated in the appendix. In this way, one obtains coefficients c_{ii}^{X} in the definition (8) which further give the formulae

$$\begin{aligned}
&\tau(E_{11}) = (q_{12}p_{12} + q_{13}p_{13}) \otimes 1 + 1 \otimes E_{11}\\
&\tau(E_{ij}) = (-q_{1j}p_{1i}) \otimes 1 + 1 \otimes E_{ij} \qquad i, j = 2, 3\\
&\tau(E_{12}) = q_{12} \otimes 1\\
&\tau(E_{13}) = q_{13} \otimes 1\\
&\tau(E_{21}) = (-q_{12}p_{12} - q_{13}p_{13})p_{12} \otimes 1 + p_{12} \otimes (E_{22} - E_{11}) + p_{13} \otimes E_{23}\\
&\tau(E_{31}) = (-q_{12}p_{12} - q_{13}p_{13})p_{13} \otimes 1 + p_{13} \otimes (E_{33} - E_{11}) + p_{12} \otimes E_{32}
\end{aligned}$$

and

$$\begin{aligned}
&\tau'(E_{11}) = (q_{12}p_{12} + q_{13}p_{13}) \otimes 1 + 1 \otimes (E_{11} + 1)\\
&\tau'(E_{ij}) = (-q_{1j}p_{1i}) \otimes 1 + 1 \otimes (E_{ij} - \tfrac{1}{2}\delta_{ij}) \qquad i, j = 2, 3\\
&\tau'(E_{12}) = q_{12} \otimes 1\\
&\tau'(E_{13}) = q_{13} \otimes 1\\
&\tau'(E_{21}) = (-q_{12}p_{12} - q_{13}p_{13})p_{12} \otimes 1 + p_{12} \otimes (E_{22} - E_{11} - \tfrac{3}{2}) + p_{13} \otimes E_{23}\\
&\tau'(E_{31}) = (-q_{12}p_{12} - q_{13}p_{13})p_{13} \otimes 1 + p_{13} \otimes (E_{33} - E_{11} - \tfrac{3}{2}) + p_{12} \otimes E_{32}.
\end{aligned}$$

According to theorems 2 and 3 these realisations are Schur realisations of the algebra gl(3, R) in $\mathrm{W}_4 \otimes \mathrm{U}(\tilde{\mathrm{g}}_0^b)$ furthermore, the realisation τ' is skew-Hermitian. On this example it is seen that the described method is also applicable to the reductive Lie algebras.

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Appendix

Using formulae (5) and (19) we obtain

$$\begin{aligned}
&E_{12}E_{12}^{k_{12}}E_{13}^{k_{13}} = E_{12}^{k_{12}+1}E_{13}^{k_{13}}\\
&E_{13}E_{12}^{k_{12}}E_{13}^{k_{13}} = E_{12}^{k_{12}}E_{13}^{k_{13}+1}\\
&E_{11}E_{12}^{k_{12}}E_{13}^{k_{13}} = k_{12}E_{12}^{k_{12}-1+1}E_{13}^{k_{13}} + k_{13}E_{12}^{k_{12}}E_{13}^{k_{13}-1+1} + E_{12}^{k_{12}}E_{13}^{k_{13}}E_{11}\\
&E_{22}E_{12}^{k_{12}}E_{13}^{k_{13}} = -k_{12}E_{12}^{k_{12}-1+1}E_{13}^{k_{13}} + E_{12}^{k_{12}}E_{13}^{k_{13}}E_{22}\\
&E_{33}E_{12}^{k_{12}}E_{13}^{k_{13}} = -k_{13}E_{12}^{k_{12}}E_{13}^{k_{13}-1+1} + E_{12}^{k_{12}}E_{13}^{k_{13}}E_{33}\\
&E_{23}E_{12}^{k_{12}}E_{13}^{k_{13}} = -k_{12}E_{12}^{k_{12}-1}E_{13}^{k_{13}+1} + E_{12}^{k_{12}}E_{13}^{k_{13}}E_{23}\\
&E_{32}E_{12}^{k_{12}}E_{13}^{k_{13}} = -k_{13}E_{12}^{k_{12}+1}E_{13}^{k_{13}-1} + E_{12}^{k_{12}}E_{13}^{k_{13}}E_{32}
\end{aligned}$$

$$E_{21}E_{12}^{k_{12}}E_{13}^{k_{13}} = k_{12}E_{12}^{k_{12}-1}E_{13}^{k_{13}}(E_{22}-E_{11}) - k_{12}(k_{12}-1)E_{12}^{k_{12}-2+1} - k_{12}k_{13}E_{12}^{k_{12}-1}E_{13}^{k_{13}-1+1} + k_{13}E_{12}^{k_{12}}E_{13}^{k_{13}-1}E_{23} + E_{12}^{k_{12}}E_{13}^{k_{13}}E_{21}$$

$$E_{31}E_{12}^{k_{12}}E_{13}^{k_{13}} = k_{12}E_{12}^{k_{12}-1}E_{13}^{k_{13}}E_{32} - k_{12}k_{13}E_{12}^{k_{12}-1+1}E_{13}^{k_{13}-1} + k_{13}E_{12}^{k_{12}}E_{13}^{k_{13}-1}(E_{33}-E_{11}) - k_{13}(k_{13}-1)E_{12}^{k_{12}}E_{13}^{k_{13}-1+1} + E_{12}^{k_{12}}E_{13}^{k_{13}}E_{31}.$$

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